Indian Statistical Institute

Final Examination 2022-2023

Analysis of Several Variables, B.Math Second Year

Time : 3 Hours   Date : 16.11.2022   Maximum Marks : 100   Instructor : Jaydeb Sarkar

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Q1. (15 marks) Let f be a C^2-function on $[0,1]^2$. Suppose $f(1,1) = f(0,1)$, $f(1,0) = 1$, and $f(0,0) = 2$. Then

$$\frac{1}{2}\Big(\int_{[0,1]^2} \frac{\partial^2 f}{\partial x\, \partial y} + \int_{[0,1]^2} \frac{\partial^2 f}{\partial y\, \partial x}\Big) = ?$$

Q2. (15 marks) Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$F(x,y) = (e^x \cos y, e^x \sin y) \qquad ((x,y) \in \mathbb{R}^2).$$

Prove that F is not injective yet it is locally invertible, and explain why this does not violate the inverse function theorem.

Q3. (20 marks) Evaluate the following integrals. Justify all steps.

$$(i) \int_0^1 \int_x^1 \Big(\sin(y^2)\Big)\, dy\, dx. \qquad (ii) \int_{-1}^1 \int_{|y|}^1 (x+y)^2\, dx\, dy.$$

Q4. (20 marks) Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a C^1-function. Suppose

$$\frac{\partial F}{\partial x}(x,y,z)\frac{\partial F}{\partial y}(x,y,z)\frac{\partial F}{\partial z}(x,y,z) \neq 0,$$

for all $(x,y,z) \in \mathbb{R}^3$. If $F(x,y,z) = 0$, then explain the meaning of $\frac{\partial x}{\partial y}$, $\frac{\partial y}{\partial z}$ and $\frac{\partial z}{\partial x}$, and prove that

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1.$$

Q5. (20 marks) (i) Let F be a C^2-vector field on an open subset $\mathcal{O} \subseteq \mathbb{R}^3$. Prove that

$$\text{div curl } F = 0.$$

(ii) Explain why the following system of partial differential equations has no C^2-solutions $P(x,y,z), Q(x,y,z)$, and $R(x,y,z)$:

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = x, \;\; \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = y, \;\; \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = z.$$

Q6. (20 marks) Verify Stokes' Theorem for the vector field $F(x,y,z) = \langle -y^2, x, z\rangle$ and the surface S obtained by intersecting $y + z = 2$ and $x^2 + y^2 \leq 1$ with standard orientation.